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THE AUSLANDER CONJECTURE FOR GROUPS LEAVING A FORM OF SIGNATURE $(n-2,2)$ INVARIANT

BY

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ABSTRACT

The Auslander conjecture claims that every affine crystallographic group Γ is virtually solvable. We prove here this conjecture for the case when the linear part of Γ is contained in the orthogonal group $O(n-2,2)$.

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1. Introduction

Let $\text{Aff}\,\mathbb{R}^n$ be the group of affine transformations of the real affine space $\mathbb{R}^n$. A subgroup Γ of $\text{Aff}\,\mathbb{R}^n$ is called properly discontinuous if $\{\gamma \in \Gamma; \gamma K \cap K \neq \emptyset\}$ is finite for every compact subset K of $\mathbb{R}^n$; and Γ is called crystallographic if Γ is properly discontinuous and the orbit space $\Gamma\backslash\mathbb{R}^n$ is compact. A subgroup Γ of $\text{Aff}\,\mathbb{R}^n$ will also be called an affine group. A long-standing conjecture of Auslander states that every affine crystallographic group Γ is virtually solvable. So far, only special cases of this conjecture have been proved; see [FG], [GrM]. For the state of the results, see [A]. The main result of this announcement deals with the following situation. Since $\text{Aff}\,\mathbb{R}^n = GL(n,\mathbb{R}) \ltimes \mathbb{R}^n$ there is a natural homomorphism ℓ: $\text{Aff}\,\mathbb{R}^n \to GL(n,\mathbb{R})$, called the linear part. Let B be a non-degenerate quadratic form on $\mathbb{R}^n$ of signature $(n-2,2)$ and let $O(B)$ be the orthogonal group of the form B.

THEOREM 1.1: *Let Γ be an affine crystallographic group with $\ell(\Gamma) \subset O(B)$. Then Γ is virtually solvable.*

In the case under consideration this result settles the Auslander conjecture completely. To put this result into perspective let us recall the following results. Let Γ be an affine crystallographic group and suppose $\ell(\Gamma) \subset O(B)$ for a non-degenerate quadratic form B of signature (p,q). Then Γ is virtually abelian if B is positive definite, i.e., $q = 0$. This is an old theorem of Bieberbach. Γ is virtually solvable if $q = 1$ [GK]. The content of Theorem 1.1 is that Γ is virtually solvable if $q = 2$. The methods of our proof are completely different from the ones used for the case $q = 1$. There are further results saying: there exists a properly discontinuous group Γ such that $\ell(\Gamma)$ is Zariski dense in $O(B)$, $(p,q) = (n, n-1)$, if n is even [AMS 3]. For every properly discontinuous group Γ the group $\ell(\Gamma)$ is not Zariski dense in $O(B)$ if $|p-q| \geq 2$ [AMS 2] or $(p,q) = (n, n-1)$ if n is odd [AMS 3]. Recently we have proved a much stronger result than in [AMS 2]

THEOREM 1.2: *Let Γ be an affine group acting properly discontinuously on an affine space V and let G be the Zariski closure of the linear part of Γ. Assume that the vector space V is a direct sum of G-invariant spaces $V = V_1 \oplus \cdots \oplus V_s$, such that on each V_i for $i = 1, \dots, s$ there exists a quadratic form B_i which is invariant under G_i, the restriction of G to V_i, that the quadratic forms $B_i, 1 \leq i \leq s$, are either positive definite or non-degenerate of signature (p_i, q_i), $|p_i - q_i| \geq 2$. Then either G is a compact group or there exists an i, $1 \leq i \leq s$, such that the group $O(B_i)$ is non-compact and G_i is a proper subgroup of $O(B_i)$.*

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2. Linear parts

Returning to the theorem of this announcement, i.e., to the case of signature $(n-2,2)$, we have to show that $\ell(\Gamma)$ is virtually solvable, since the kernel of ℓ is abelian, or equivalently that the Zariski closure of $\ell(\Gamma)$ is virtually solvable. The proof is done by contradiction, so we will assume from this point on that the Zariski closure of $\ell(\Gamma)$ is not solvable. We can also assume that it is connected. We may assume furthermore that the dimension n of our affine space is minimal among the counterexamples to our theorem. Let V be a vector space, B a quadratic form on V, and $O(B)$ the orthogonal group of the form B. We will say that a connected simple subgroup H of $O(B)$ is a standard subgroup if V is a direct sum of H-invariant, B-orthogonal subspaces W_1 and W_2 such that the natural homomorphism $\pi\colon H \longrightarrow O(B_1)$ is onto, where B_1 is the restriction of B to W_1, and $h|_{W_2} = id$ for every $h \in H$.

LEMMA 2.1: *Assume that B is a quadratic form of signature $(n-2,2)$ and H a connected simple subgroup of $O(B)$ and $rank_{\mathbb{R}} H = 2$; then H is a standard subgroup of $O(B)$.*

Proof: Let $\mathfrak{g}$ be the Lie algebra of $O(B)$. We will use the following matrix realization of the Lie algebra $\mathfrak{g}$. Let J be the following matrix:

$$J = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & I_{n-4} & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{g}=\{A \in M_n(\mathbb{R}), AJ = JA^t\}$ [OV]. There exists a maximal $\mathbb{R}$- split torus

T in $O(B)$, whose Lie algebra $\mathfrak{t}$ is the set $\mathfrak{t} = \{t \in M_n(\mathbb{R}), \varepsilon_1 \in \mathbb{R}, \varepsilon_2 \in \mathbb{R}\}$ where

$$t = \begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 & 0 \\ 0 & \varepsilon_2 & \dots & 0 & 0 \\ \vdots & \vdots & 0_{n-4} & \vdots & \vdots \\ 0 & 0 & \dots & -\varepsilon_2 & 0 \\ 0 & 0 & \dots & 0 & -\varepsilon_1 \end{pmatrix}.$$

Therefore, all positive roots are the following: $\alpha = \varepsilon_1$, $\beta = \varepsilon_2$, $\alpha + \beta = \varepsilon_1 + \varepsilon_2$, $\alpha - \beta = \varepsilon_1 - \varepsilon_2$. The dimensions of the corresponding root spaces are as follows: $\dim V_\alpha = \dim V_\beta = n-2$, $\dim V_{\alpha+\beta} = \dim V_{\alpha-\beta} = 1$.

We can assume that $T \le H$. Let G_0 be the smallest connected simple subgroup of $O(B)$, containing T. Then $G_0 \le H$. Let $\mathfrak{g}_0$ be the Lie algebra of G_0. Then the simple algebra $\mathfrak{g}_0$ contains $\mathfrak{t}$ and therefore all the following root spaces $V_{\alpha+\beta}, V_{-(\alpha+\beta)}, V_{\alpha-\beta}, V_{-(\alpha-\beta)}$. Let U^+ be the sum $U^+ = V_\alpha + V_\beta$ and $U^- = V_{-\alpha} + V_{-\beta}$. Let $\mathfrak{h}$ be the Lie algebra of H; then $\mathfrak{h} \cap U^+ \neq \{0\}$. This intersection is T invariant, therefore $\mathfrak{h} \cap V_\alpha \neq \{0\}$ and $\mathfrak{h} \cap V_\beta \neq \{0\}$. There exists an element w_1 in the Weyl group of G_0 such that $w_1 V_\alpha {w_1}^{-1} = V_\beta$. Therefore, $w_1(V_\alpha \cap \mathfrak{h}){w_1}^{-1} = V_\beta \cap \mathfrak{h}$. Let K be the centralizer of T, $K = Z_{O(B)}(T)$. This group acts transitively on U^+ and U^-. There exists an element w_2 in the Weyl group of G_0 such that $w_2 U^+ {w_2}^{-1} = U^-$. Therefore, one can find a standard subgroup $\widehat{H}$ such that if $\hat{\mathfrak{h}}$ is the Lie algebra of $\widehat{H}$, then $\hat{\mathfrak{h}} \cap U^+ = \mathfrak{h} \cap U^+$ and $\hat{\mathfrak{h}} \cap U^- = \mathfrak{h} \cap U^-$. Thus, the sets of unipotent elements in H and $\widehat{H}$ are the same and hence $H = \widehat{H}$, since both of them are generated by their unipotent elements. ∎

Remark 2.2: Actually, using the same idea, one can prove that if the signature of a quadratic form B is $(n-k, k)$ and H is a connected simple subgroup in $O(B)$ of real rank k, then H is standard.

LEMMA 2.3: *Let G be the Zariski closure of $\ell(\Gamma)$. Then G is a reductive group.*

Proof: Let U be the unipotent radical of G and let S be a semisimple part of G. If all simple connected subgroups of S have real rank at most 1, then (see [A]) Γ is virtually solvable. Therefore, there is one connected simple normal subgroup S_1 of S with real rank 2. From the previous lemma it follows that there are two S-invariant B-orthogonal subspaces W_1 and W_2, such that $S|_{W_1} = O(B|_{W_1})$, and the restriction $B|_{W_2}$ is a positive definite quadratic form. Let $V_0 = \{v \in V \mid tv = v \text{ for all } t \in U\}$. This subspace is G-invariant and it is easy to see that either $V_0 \subseteq W_2$, which is impossible, or $V_0 = V$. Thus $U = \{1\}$. ∎

Let us summarize:

LEMMA 2.4: *Let G be the Zariski closure of $\ell(\Gamma)$. Then there are two non-zero G-invariant B-orthogonal vector subspaces W_1 and W_2 in $\mathbb{R}^n$ such that*

(1) $W_1 \oplus W_2 = \mathbb{R}^n$,
(2) *the restriction $B|_{W_1}$ of the form B_1 to W_1 has signature $(m-2, 2)$, where $m = dimW_1$ and $G|_{W_1} = O(B_1)$,*
(3) *the restriction $B|W_2$ is positive definite,*
(4) *G is the natural direct product of $O(B_1)$ and a compact group K.*

Let B be a quadratic form on $\mathbb{R}^n$ of signature $(p,q), q \leq p$, $p+q=n$ and $H_B = SO(B)$. Let us first recall the following definitions from [AMS 1]. Assume that g is a semisimple element of H_B. Then the space $\mathbb{R}^n$ can be decomposed into a direct sum of three subspaces $A^+(g), A^-(g), A^0(g)$ determined by the condition that all eigenvalues of the restriction $g \mid A^+(g)$ (resp. $g \mid A^-(g)$, $g \mid A^0(g)$) are of modulus more than 1 (resp. less than 1, equal to 1). An element g of H_B is called **hyperbolic** if $\dim A^0(g) = p - q$. Let B_g be the restriction of the quadratic form B to $A^0(g)$. Then for every hyperbolic element g of H_B the form B_g is positive definite. Let π_g be the natural projection $\pi_g\colon \mathbb{R}^n \longrightarrow A^0(g)$ parallel to the subspace $A^+(g) \oplus A^-(g)$. Put $D^+(g) = A^+(g) \oplus A^0(g)$ and $D^-(g) = A^-(g) \oplus A^0(g)$; then obviously $D^+(g) \cap D^-(g) = A^0(g)$. Let g be a hyperbolic element and let $s^+(g) = \max\{|\lambda_g|, \lambda_g$ an eigenvalue of g of modulus $< 1\}$. Let $s^-(g) = s^+(g^{-1})$ and $s(g) = \max\{s^+(g), s^-(g)\}$. We will fix the standard scalar product on $\mathbb{R}^n$ and denote by $| \,\boldsymbol{\cdot}\, |$ and d the corresponding norm and metric on $\mathbb{R}^n$. This metric induces in the standard way a metric $\widehat{d}$ on the projective space $\mathbb{P}\mathbb{R}^n$. A hyperbolic element $g \in H_B$ is called ε-**hyperbolic** if $\widehat{d}(A^+(g), A^-(g)) \geq \varepsilon$. Two hyperbolic elements g and h are called ε-**transversal** if

$$\widehat{d}(A^+(g), A^-(h)) \geq \varepsilon \quad \text{and} \quad \widehat{d}(A^-(g), A^+(h)) \geq \varepsilon.$$

Put $o(g) = g \mid A^0(g)$. Let g and h be ε-hyperbolic, ε-transversal elements in H_B. We will now define an isometry $\rho\colon A^0(h) \longrightarrow A^0(g)$ as follows. Let $A_0(g,h) = D^)(g) \cap D^+(h)$. Then we have two projections: $A^0(h) \longrightarrow A_0(g,h)$, which is a projection parallel to $A^+(h)$, and $A_0(g,h) \longrightarrow A^0(g)$, which is a projection parallel to $A^-(g)$. Then we define ρ as their composition. Let $g_0, g_1, \ldots, g_n$ be ε-hyperbolic pairwise ε-transversal elements in H_B. Then, as above, for every pair (g_i, g_{i+1}) we have an isometry $\rho_{i+1}\colon A^0(g_{i+1}) \longrightarrow A^0(g_i)$. Put $\pi_i = \rho_1 \cdots \rho_i$. Then $\hat{o}(g_i) = \pi_i o(g_i) {\pi_i}^{-1}$ is an orthogonal transformation of $A^0(h_0)$. Let $\ell = (\ell_0, \ell_1, \ldots, \ell_{n-1}) \in \mathbb{N}^n$, $g^\ell = g_0^{\ell_0} g_1^{\ell_1} \cdots g_{n-1}^{\ell_{n-1}}$ and $o_\ell = \hat{o}(g_0)^{\ell_0} \hat{o}(g_1)^{\ell_1} \cdots \hat{o}(g_{n-1})^{\ell_{n-1}}$. Clearly $\hat{o}(g_i)^{\ell_i} = \hat{o}(g_i^{\ell_i})$.

An important role in our proof of the next lemma is played by the main result of the recent paper [PR] by G. Prasad and A. S. Rapinchuk.

LEMMA 2.5: *Let Γ be a Zariski dense subgroup in H_B and g and h be hyperbolic transversal elements of Γ. Put $g_0 = g$. There exist a positive real number ε and elements $g_i \in \Gamma$, for $i = 1, \dots, n-1$, such that with $g_n = h$:*

(1) *$g_0, g_1, \dots, g_n$ are ε-hyperbolic, pairwise ε-transversal elements.*

(2) *The set $\{o_\ell\}_{\ell \in \mathbb{N}^n}$ is dense in the connected component of the group $O(B_{g_0})$.*

Proof: The group Γ is a Zariski dense subgroup of $O(B)$ and, according to [AMS 3], for every n there are hyperbolic elements $g_i, i = 1, \dots, n-1$, such that $g_0, g_1, \dots, g_n$ are ε-hyperbolic, pairwise ε-transversal elements, for suitable ε. Let us now explain how to show that (2) is true.

It is enough to show that for every hyperbolic element g there exist hyperbolic, pairwise transversal elements $g_1, \dots, g_n$ such that the closure of the set $\{o_\ell\}_{\ell \in \mathbb{N}^n}$ contains a Zariski open subset. Assume that we have chosen elements $g_i, i = 1, \dots, n-1$ such that the closure O of the set $\{o_\ell\}_{\ell \in \mathbb{N}^n}$ has maximal possible dimension. The set O is constructible and therefore there exist Zariski closed subsets K_i and Zariski open subsets U_i of $O(B_{g_0})$, $1 \le i \le m$ such that $O = \bigcup_{i=1}^m (K_i \cap U_i)$. Assume that for every i, $i = 1, \dots, m$, K_i is a proper subset of $O(B_{h_0})$. Let

$$S_1 = \{\gamma | \gamma \in \Gamma, \gamma \text{ is hyperbolic and } \mathbb{R}\text{-irreducible element}\}.$$

This set is nonempty and open in Γ (see [PR]). Let

$$S_2 = \{\gamma | \gamma \in \Gamma, \gamma \text{ and } g_{n-1} \text{ are transversal}\}.$$

This set is also open and nonempty. Therefore, the same is true for $S = S_1 \cap S_2$. Let $\gamma \in S$ and let ρ: $A^0(\gamma) \longrightarrow A^0(g_{n-1})$ be the projection we defined above. Put $\pi_{n+1} = \pi_n \rho$. Let us now take an element t from H_B and consider the element $\gamma(t) = t\gamma t^{-1}$. Define T_γ to be the set of all regular elements $t \in H_B$ such that $A^+(\gamma) = A^+(t)$, $A^-(\gamma) = A^-(t)$ and therefore $A^0(\gamma) = A^0(t)$. It is easy to see that the set

$$T_1 = \{o(t)o(\gamma)^n o(t^{-1}) | n \in \mathbb{N}, t \in T_\gamma\}$$

is dense in $O(B_\gamma)$, because γ is $\mathbb{R}$-irreducible. Therefore, the set

$$T = \{t \in H_B | \overline{\{\hat{o}(\gamma(t))^n\}_{n \in \mathbb{N}}} \subsetneq K \text{ where } K = \bigcup_{i=1}^m K_i\}$$

is open and nonempty. Let $t \in T \cap S$. Let us take an element $g_n = t\gamma t^{-1}$. If we add g_n to the chosen set g_i, $i = 1, \dots, n-1$ we increase the dimension of the set $\{o_\ell\}_{\ell \in \mathbb{N}^{n+1}}$, which is impossible. Therefore, $K_i = O(B_{g_0})$ for some i. Then the closure of the set $\{o_\ell\}_{\ell \in \mathbb{N}^n}$ contains Zariski open subsets. ∎

Let $\mathfrak{O}(B)$ be the following set:

$$\mathfrak{O}(B) = \{(W,g);\ W \text{ a maximal } B\text{-anisotropic subspace of } \mathbb{R}^n,$$
$$B_W \text{ the restriction of } B \text{ on } W, \text{ and } g \in O(B_W)\}.$$

We will say a sequence $\{X_n\}_{n \in \mathbb{N}}$, $X_n = (W_n, g_n) \in \mathfrak{O}(B)$, converges to $X = (W, g), X \in \mathfrak{O}(B)$, if

(1) $\hat{d}(W_n, W) \longrightarrow 0$ when $n \longrightarrow \infty$;
(2) for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for every pair (a,b) of vectors with $a \in W, |\,a| = 1$, and $b \in W_n, |\,b| = 1$, we have

$$|\,a - b| - \varepsilon \le |\,g(a) - g_n(b)| \le |\,a - b| + \varepsilon, \quad \text{for all } n > N.$$

We will then write $X_n \rightrightarrows X$.

For any hyperbolic element g of $O(B)$, put $X_g = (A^0(g), o(g)) \in \mathfrak{O}(B)$. Let $\mathfrak{O}(\Gamma) = \{X_g | g \in \Gamma,\ g \text{ hyperbolic}\}$ and $\mathfrak{O}^\varepsilon(\Gamma) = \{X_g | g \in \Gamma,\ g\ \varepsilon\text{-hyperbolic}\}$.

Lemma 2.6: *Let Γ be a Zariski dense subgroup in $O(B)$ and let $\overline{\mathfrak{O}(\Gamma)}$ be the closure of $\mathfrak{O}(\Gamma)$ in $\mathfrak{O}(B)$ and $\overline{\mathfrak{O}^\varepsilon(\Gamma)}$ be the closure of $\mathfrak{O}^\varepsilon(\Gamma)$ in $\mathfrak{O}(B)$.*

(1) *If $(W,g) \in \overline{\mathfrak{O}(\Gamma)}$ and $W = A^0(\gamma)$ for some hyperbolic element $\gamma \in \Gamma$, then $(W,g) \in \overline{\mathfrak{O}(\Gamma)}$ for every $g \in O(B_W)$.*
(2) *If $(W,g) \in \overline{\mathfrak{O}^\varepsilon(\Gamma)}$, then $(W,g) \in \overline{\mathfrak{O}^\varepsilon(\Gamma)}$ for every $g \in O(B_W)$.*

3. Affine groups and sketch of the proof

Let A_B be the subgroup of the affine group Aff $\mathbb{R}^n$ consisting of those elements g whose linear part belongs to H_B. Recall that $H_B = SO(B)$, where B is a non-degenerate quadratic form and $O(B)$ is an orthogonal group of the form B. We will call $g \in A_B$ hyperbolic if the linear part $l(g)$ is hyperbolic. Let us remember that for a linear hyperbolic transformation $l(g)$, we have defined three vector spaces $A^+(l(g))$, $A^-(l(g))$, $A^0(l(g))$ determined by the condition that all eigenvalues of the restriction $g \mid A^+(l(g))$ (resp. $g \mid A^-(l(g))$, $g \mid A^0(l(g))$) are of modulus more than 1 (resp. less than 1, equal to 1). For an element $g \in A_B$ we will still write $A^+(g)$ ($A^-(g)$, $A^0(g)$) instead of $A^+(l(g))$ ($A^-(l(g))$, $A^0(l(g))$). Let now g be a hyperbolic element of A_B for which there exists a unique g-invariant line L_g. That will be the case if, for example, g is a hyperbolic element of Γ and Γ acts properly discontinuous on $\mathbb{R}^n$. In this case the restriction of g to L_g is a parallel translation by a vector t_g. Note that $t_g \in A^0(g)$, so $B_g(t_g,t_g) > 0$. Let $v^0(g) = t_g/B_g(t_g,t_g)$, and $B_g(v^0(g),v^0(g)) = 1$, so

$$B(gx - x, v^0(g)) = t_g$$

for any point $x \in \mathbb{R}^n$. Let us now define the following affine subspaces: $E_g^+ = D^+(g) + L_g$, $E_g^- = D^-(g) + L_g$ and $C_g = E_g^+ \cap E^-{}_g$. It is clear that $L_g \subseteqq C_g$.

We will also use the notation π_g for the natural projection $\pi_g\colon \mathbb{R}^n \longrightarrow C_g$ parallel to $A^+(g) \oplus A^-(g)$.

LEMMA 3.1: *Let g_0, $h_1,\ldots,h_m$ be ε-hyperbolic, pairwise ε-transversal elements. Let H be the group generated by $h_1,\ldots,h_m$ and let $g_h = g_0h$. We will fix a point $q \in \mathbb{R}^n$ and put $c_{g_h} = d(q, C_{g_h})$. Let $s = \max\{s(g_0), s(h_1),\ldots,s(h_m)\}$. Then there exist ε, $a = a(\varepsilon)$, $c = c(\varepsilon)$ such that for $s \le a$*

(1) *g_h is $\varepsilon/2$-regular for all $h \in H$,*
(2) *$c_{g_h} \le c$ for all $h \in H$.*

Let $\mathfrak{D}_a(B)$ be the set of all (X,v) where $X = (W,g) \in \mathfrak{D}(B)$, $v \in W$ and $B(v,v) = 1$. The set $\mathfrak{D}_a(B)$ contains information about the affine transformation g, not only about its linear part $\ell(g)$, therefore we add the index a to $\mathfrak{D}(B)$. We will say that the sequence $\{(X_n,v_n)\}_{n\in\mathbb{N}}$, $(X_n,v_n) \in \mathfrak{D}_a(B)$ converges to $(X,v) \in \mathfrak{D}_a(B)$ if $X_n \rightrightarrows X$ and $v_n \to v$. Let

$$\mathfrak{D}_a^\varepsilon(\Gamma) = \{(X,v) \in \mathfrak{D}_a(B); X \in \mathfrak{D}^\varepsilon(\Gamma), v \in W, B(v,v) = 1\}.$$

The following results play a central role in our proof. We will prove it under the following assumptions about an affine group Γ.

(I) B is a non-degenerate quadratic form of signature (p,q), where $p \ge q$, and one of the following two conditions holds: $p - q \ge 2$ or $q \le 2$.
(II) $\ell(\Gamma)$ is Zariski dense in $O(B)$.
(III) Every hyperbolic element $\gamma \in \Gamma$ has no fixed point.

Note that we do not assume that Γ is properly discontinuous.

PROPOSITION 3.2: *Let Γ be a subgroup of $\operatorname{Aff}\mathbb{R}^n$ with the properties* (I)–(III) *above. Then there exist elements $X_1,\ldots,X_m,Y_1,\ldots,Y_m$ with the following properties:*

(1) *$X_i \in \overline{\mathfrak{D}^\varepsilon{}_a(\Gamma)}$ for $i = 1,\ldots,m$ and $Y_i \in \overline{\mathfrak{D}^\varepsilon{}_a(\Gamma)}$ for $i = 1,\ldots,m$.*
(2) *For all $i = 1,\ldots,m$ we have $X_i = (W,g_i,v_i)$ and $Y_i = (W,h_i,-v_i)$.*
(3) *$\{v_1,v_2,\ldots,v_m\}$ forms a basis of W.*

Based on Proposition 3.2 we prove

PROPOSITION 3.3: *Let Γ be a subgroup of* $\mathrm{Aff}\,\mathbb{R}^n$ *with the properties* (I)–(III) *above. Then there is an* $\varepsilon > 0$, *a set of* ε*-regular,* ε*-transversal elements* $\gamma_1, \ldots, \gamma_s$ *in* Γ, *a compact subset* K *of* $\mathbb{R}^n$ *and constants* $C_0 = C_0(\varepsilon)$, $C_1 = C_1(\varepsilon)$, $r = r(\varepsilon) > 1$ *and* $q = q(\varepsilon) < 1$ *such that:*

(1) $\gamma_1, \ldots, \gamma_s$ *are free generators of a free group* $\Gamma^* = \langle \gamma_1, \ldots, \gamma_s \rangle$.

(2) *If* γ *is an* ε*-regular element in* Γ *such that* $\{\gamma, \gamma_1, \ldots, \gamma_s\}$ *are pairwise* ε*-transversal elements and* $r(\gamma) \geq r$, *then* $\gamma, \gamma_1, \ldots, \gamma_s$ *are free generators of a free group* $\widehat{\Gamma}$.

(3) *If* γ *is as in* (2) *and* $d^B_\gamma(K) > C_0$ *and* $r(\gamma) \geq r$, *then there are a number* $t \in \{1, \ldots, s\}$ *and a positive integer* $m \leq d^B_\gamma(K) \cdot C_1$ *such that for* $\widehat{\gamma} = \gamma_t^m \cdot \gamma$ *we have* $d^B_{\widehat{\gamma}}(K) \leq q \cdot d^B_\gamma(K)$.

Let us explain the connection between these two important propositions. Assume that there exist hyperbolic, pairwise transversal elements $g_1, \ldots, g_m$ in Γ and $h_1, \ldots, h_m$ in Γ such that if $X_i = (A^0(g_i), o(g_i))$, $v_i = v^0(g_i)$, $Y_i = (A^0(h_i), o(h_i))$, $v_i = v^0(h_i)$ satisfy conditions (1)–(3) of Proposition 3.2. Then using Lemma 3.1 one can see that such elements satisfy Proposition 3.3. By Proposition 3.2, we can find such elements $X_1, \ldots, X_s, Y_1, \ldots, Y_s$ in the closure of $\mathfrak{D}^\varepsilon_a(\Gamma)$. The content of Proposition 3.3 is to show that if we take elements from Γ quite close to the elements $X_1, \ldots, X_s, Y_1, \ldots, Y_s$, then they will also satisfy Proposition 3.3.

From these propositions follows

COROLLARY 3.4: *With notations and assumptions as in the proposition, let* w *be the word length on* $\Gamma^* = \langle \gamma_1, \ldots, \gamma_s \rangle$. *Then there is a constant* $C = C(\varepsilon)$ *and an element* $\gamma^* \in \Gamma^*$ *such that* $w(\gamma^*) \leq (d^B_\gamma(K) \cdot C)^2$ *and* $d^B_{\gamma^* \gamma}(K) \leq C_0$.

The idea of the proof of Theorem 1 is as follows. We decompose every $\gamma \in \Gamma$ into two components, along W_1 and along W_2. The proposition shows that for the W_1-component there is a coming-back effect. The corollary shows furthermore that one has control over the word length of the elements involved. It follows that exponentially many elements have the property that their W_1-component returns near the starting point. Considering their W_2-component, one plays this exponential growth of the number of elements against the polynomial growth of the volume to see that there are infinitely many elements whose W_2-component returns close to the starting point. Since one has control over the E^0_γ-spaces, we conclude a contradiction to proper discontinuity.

We will now explain the main lines of the proof in more detail. Assume Γ is not virtually solvable. Then the Zariski closure G of $\ell(\Gamma)$ is semisimple, and there is a decomposition $\mathbb{R}^n = W_1 \oplus W_2$ where W_i, $i = 1, 2$, are G-invariant, W_1 is an irreducible G-vector space, $B_1 = B|W_1$ has signature $(n_1 - 2, 2)$ and $B_2 = B|W_2$ is positive definite. We have a projection π_1 of the affine space $\mathbb{R}^n$ to the affine space $A_1 = \mathbb{R}^n/W_2$ along W_2, and hence an induced homomorphism $\pi_1\colon \Gamma \to \operatorname{Aff} A_1$ and similarly for $\pi_2\colon \mathbb{R}^n \to \mathbb{A}_2 := \mathbb{R}^n/W_1$. As a first step we show that the representation of G on W_1 has property $(*)$ and $\pi_1(\Gamma)$ is Zariski dense in $O(B_1)$. We can thus apply Proposition 3.3 to the group $\Gamma_1 = \pi_1(\Gamma)$ in the following way. We find elements $\gamma_1, \ldots, \gamma_s$ in Γ such that the elements $\pi_1(\gamma_1), \ldots, \pi_1(\gamma_s)$ are as in Proposition 3.3. As in [AMS 1, AMS 3] we can then choose two elements $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in Γ both regular and such that the elements $\pi_1(\tilde{\gamma}_i)$, $\pi_1(\gamma_j)$, $i = 1, 2$, $j = 1, \ldots, s$ are pairwise transversal. Let $\varepsilon_1 < \varepsilon$ be chosen so that these elements are ε_1-regular and pairwise ε_1-transversal. Then there is a natural number N such that for every $\gamma \in \langle \tilde{\gamma}_1{}^N, \tilde{\gamma}_2{}^N \rangle$ we have

(a) $\pi_1(\gamma)$ is $\varepsilon_{1/2}$-regular and $r(\pi_1(\gamma)) \geq r(\varepsilon_1/2)$,

(b) $\{\pi_1(\gamma), \pi_1(\gamma_i), i = 1, \ldots, s\}$ are pairwise $\varepsilon_1/2$-transversal.

By changing notation we will assume that $N = 1$ and put $\varepsilon = \varepsilon_1/2$. Let w be the word metric on $\widehat{\Gamma} = \langle \gamma_1, \ldots, \gamma_s, \tilde{\gamma}_1, \tilde{\gamma}_2 \rangle$ corresponding to these generators. It is easy to check that for any two compact subsets $K_1 \subset A_1$ and $K_2 \subset A_2$ we have

(1) $d^{B_1}_{\pi_1(\gamma)}(K_1) \ll w(\gamma)$,

(2) $d^{B_2}_{\pi_2(\gamma)}(K_2) \ll w(\gamma)$.

Now let $S(M) = \{\gamma \in \langle \tilde{\gamma}_1, \tilde{\gamma}_2 \rangle;\ w(\gamma) \leq M\}$. Then $|S(M)| \geq 3^M - 1$. For every $\gamma \in S(M)$ there is an element $\gamma^* \in \Gamma^*$ such that

$$w(\gamma^*) \leq (d^{B_1}_{\pi_1(\gamma)}(K_1) \cdot C)^2 \quad \text{and} \quad d^{B_1}_{\pi_1(\gamma^* \gamma)}(K_1) \leq C_0,$$

by Lemma 3.1 It is not difficult to see that $w(\gamma^* \gamma) \ll M^2$. Therefore, if we put $T(M) = \{\gamma \in \widehat{\Gamma}; d^{B_1}_{\pi_1(\gamma)}(K_1) \leq C_0 \text{ and } w(\gamma) \leq C_3 \cdot M^2\}$ for an appropriately chosen constant C_3, we have

(3) $|T(M)| \geq 3^M - 1$.

For $p \in A$ and $\gamma \in T(M)$ we can conclude from (1) and (2) that γp is in the ball of radius $C_3 M^2$ around p. The volume of this ball is $\ll M^{2 \cdot \dim A_2}$ whereas $|T(M)|$ grows exponentially with M. Hence for every $\delta > 0$ the number of elements of $P(M) = \{(\gamma_1, \gamma_2) \in T(M) \times T(M),\ B_2(\pi_2\gamma_1(p) - \pi_2\gamma_2(p)) \leq \delta\}$ goes to infinity when $|M|$ goes to infinity. Since, for an appropriately chosen point $p \in \mathbb{R}^n$, the Euclidean distance $d(\gamma(p), p)$ is bounded for $\gamma = \gamma_1 \gamma_2^{-1}$ and $(\gamma_1, \gamma_2) \in P(M)$, it follows that Γ does not act properly discontinuously. This proves the theorem.

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